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Geometric Properties of Solutions of a Class of Ordinary Linear Differential Equations

Hitoshi Saitoh

Department of Mathematics, Gunma National College of Technology
Maebashi, Gunma 371-8530, Japan
saitoh@nat.gunma-ct.ac.jp

Abstract—The main object of this paper is to investigate several geometric properties of the solutions of the following second-order linear differential equation:

$$w''(z)+p(z)w(z)=0,$$

where the function $p(z)$ is analytic in the open unit disk $\mathbb{U}$. Relevant connections of the results presented in this paper with those given earlier by, for example, Robertson, Miller and Saitoh are also considered.

Keywords—Starlike functions, Strongly starlike functions, Close-to-convex functions, Differential equations, Univalent function.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C}\text{ and }|z|<1\}.$$

Also, let S,S^* and $S^*(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent, starlike with respect to the origin, and starlike of order α in $\mathbb{U}$ $(0\leq\alpha<1)$. Thus, by definition, we have (see, for detail, [1] [6] [8]).

$$S^*(\alpha):=\left\{f:f\in\mathcal{A}\text{ and }\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ (z\in\mathbb{U};\ 0\leq\alpha<1)\right\} \tag{1.2}$$

and

$$S^*:=S^*(0). \tag{1.3}$$

Furthermore, $SS^*(\beta)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are strongly starlike of order β in $\mathbb{U}$ $(0 < \beta \leq 1)$. By definition, we have

$$SS^*(\beta) := \left\{ f : f \in \mathcal{A} \text{ and } \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leq \frac{\pi}{2}\beta \ \ (z \in \mathbb{U};\ 0 < \beta \leq 1) \right\}. \tag{1.4}$$

For functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ $(z \in \mathbb{U})$, we define the Schwarzian derivative of $f(z)$ by

$$S(f,z) := \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2. \tag{1.5}$$

We begin by recalling the following result of Miller [3].

Theorem A. (See [3]) Let the function $p(z)$ be analytic in $\mathbb{U}$ with

$$|zp(z)| < 1 \qquad (z \in \mathbb{U}).$$

Also, let $v(z)$ denote the unique solution of the following initial-value problem:

$$v''(z) + p(z)v(z) = 0 \qquad (v(0) = 0;\ v'(0) = 1) \tag{1.6}$$

in $\mathbb{U}$. Then

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1 \qquad (z \in \mathbb{U}) \tag{1.7}$$

and $v(z)$ is a starlike conformal map of the unit disk $\mathbb{U}$.

Theorem A is related rather closely to some earlier results of Nehari [5] and Robertson [9], which we recall here as Theorem B and C below.

Theorem B. (See [5]) If $f \in \mathcal{A}$ satisfies the following inequality involving its Schwarzian derivative defined by (1.5):

$$|S(f,z)| \leq \frac{\pi^2}{2} \qquad (z \in \mathbb{U}), \tag{1.8}$$

then $f \in S$. The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{e^{i\pi z} - 1}{i\pi}. \tag{1.9}$$

Theorem C. (See [9]) Let $zp(z)$ be analytic in $\mathbb{U}$ and

$$\text{Re}\left\{ z^2 p(z) \right\} \leq \frac{\pi^2}{4}|z|^2 \qquad (z \in \mathbb{U}). \tag{1.10}$$

Then the unique solution $W = W(z)$ of the following initial-value problem:

$$W''(z) + p(z)W(z) = 0 \qquad (W(0) = 0;\ W'(0) = 1) \tag{1.11}$$

is univalent and starlike in $\mathbb{U}$.The constant $\frac{\pi^2}{4}$ in inequlity (1.10) is the best possible.

Remark 1. By putting

$$p(z) = \frac{1}{2}S(f,z) \qquad (z \in \mathbb{U}) \tag{1.12}$$

and using (1.8), we obtain inequality (1.10).Obviously, therefore, the hypothesis in Theorem B is stronger than that in Theorem C.

2 A class of bounded functions

Let $\mathcal{B}_J$ denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

analytic in $\mathbb{U}$, for which

$$|w(z)| < J \qquad (z \in \mathbb{U};\ J > 0). \tag{2.2}$$

If $g(z) \in \mathcal{B}_J$, then we can show that the function $w(z)$ defined by

$$w(z) := z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt \tag{2.3}$$

is also in the class $\mathcal{B}_J$. Thus, in terms of derivatives, we have

$$\left|\frac{1}{2}w(z) + zw'(z)\right| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}). \tag{2.4}$$

Furthermore, by setting

$$h(u, v) := \frac{1}{2}u + v, \tag{2.5}$$

we can rewrite (2.4) in the form:

$$|h(w(z), zw'(z))| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}). \tag{2.6}$$

In this section, we show that implication (2.6) holds true for functions $h(u, v)$ in the class $\mathcal{H}_J$ given by Definition 1 below (see also [4]).

Definition 1. Let $\mathcal{H}_J$ be the class of complex functions $h(u, v)$ satisfying each of the following conditions:

(i) $h(u, v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$;

(ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| < J \quad (J > 0)$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \leq J$ whenever $(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D} \quad (\theta \in \mathbb{R};\ K \geq J > 0)$.

Example 1. It is easily seen that the function

$$h(u, v) = \gamma u + v \qquad (\mathrm{Re}(\gamma) \geq 0;\ \mathbb{D} = \mathbb{C} \times \mathbb{C}) \tag{2.7}$$

is in the class $\mathcal{H}_J$.

Definition 2. Let $h \in \mathcal{H}_J$ with the corresponding domain $\mathbb{D}$. We denote by $\mathcal{B}_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in $\mathbb{U}$ and satisfy each of the following conditions:

(i) $(w(z), zw'(z)) \in \mathbb{D}$;

(ii) $|h(w(z), zw(z))| < J \qquad (z \in \mathbb{U}; J > 0)$. vspace1mm
The function class $\mathcal{B}_J(h)$ is not empty. Indeed, for any given function $h \in \mathcal{H}_J$, we have

$$w(z) = c_1 z \in \mathcal{B}_J(h), \tag{2.8}$$

for sufficiently small $|c_1|$ depending on h.

Theorem D. (See [10]) For any $h \in \mathcal{H}_J$,

$$\mathcal{B}_J(h) \subset \mathcal{B}_J \qquad (h \in \mathcal{H}_J;\ J > 0)$$

Remark 2. Theorem D show that, if $h \in \mathcal{H}_J$ (with the corresponding domain $\mathbb{D}$) and if $w(z)$, given by (2.1), is analytic in $\mathbb{U}$ and

$$(w(z), zw'(z)) \in \mathbb{D},$$

them the implication (2.4) holds true.

Theorem D leads us immediately to the following result, which was also given by [10].
Theorem E. (See [10]) Let $h \in \mathcal{H}_J$ and let the function $b(z)$ be analytic in $\mathbb{U}$ with

$$|b(z)| < J \qquad (z \in \mathbb{U};\ J > 0).$$

If the following initial-value problem:

$$h(w(z), zw'(z)) = b(z) \qquad (w(0) = 0) \tag{2.9}$$

has a solution $w(z)$ analytic in $\mathbb{U}$, then

$$|w(z)| < J \qquad (z \in \mathbb{U};\ J > 0) \tag{2.10}$$

Recently, using Theorem E, we prove the following Theorem F and Theorem G.
Theorem F. (See [10]) Let $a(z)$ and $b(z)$ be analytic in $\mathbb{U}$ with

$$\left| z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right) \right| < \frac{1}{2} \tag{2.11}$$

and

$$|a(z)| \leq 1. \tag{2.12}$$

Let $w(z)$ denote the solution of the initial-value problem:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \qquad (w(0) = 0;\ w'(0) = 1) \tag{2.13}$$

in $\mathbb{U}$. Then $w(z)$ is starlike in $\mathbb{U}$.

Theorem G. (See [7]) Let the functions $a(z)$ and $b(z)$ be analytic in $\mathbb{U}$ with

$$\left| z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right) \right| < J \qquad (z \in \mathbb{U};\ J > 0) \tag{2.14}$$

and

$$\text{Re}\{za(z)\} > -2J \qquad (z \in \mathbb{U};\ J > 0). \tag{2.15}$$

Also, let $w(z)$ denote the solution of the initial-value problem (2.13) in $\mathbb{U}$. Then

$$1 - J - \frac{1}{2}\text{Re}\{za(z)\} < \text{Re}\left\{\frac{zw'(z)}{w(z)}\right\} < 1 + J - \frac{1}{2}\text{Re}\{za(z)\} \qquad (z \in \mathbb{U};\ J > 0). \tag{2.16}$$

Example 2. Let $a(z) = 1,\ b(z) = 0$ in Theorem F, then the solution of

$$w''(z) + w'(z) = 0 \tag{2.17}$$

is $w(z) = 1 - e^{-z} \in S^*$.

Example 3. Let $a(z) = -2Jz$ and $b(z) = J^2z^2$ in Theorem G ($J > 0$), then the solution of the following initial-value problem:

$$w''(z) - 2Jzw'(z) + J^2z^2w(z) = 0 \qquad (w(0) = 0;\ w'(0) = 1) \tag{2.18}$$

is given by

$$w(z) = \frac{1}{\sqrt{J}}\exp\left(\frac{1}{2}Jz^2\right)\sin(\sqrt{J}z). \tag{2.19}$$

In this case, if the further assume that $0 < J \leq \frac{1}{2}$, then

$$w(z) \in S^*(1 - 2J) \qquad \left(0 < J \leq \frac{1}{2}\right),$$

so that, in particular, we have

$$J = \frac{1}{2} : w(z) = \sqrt{2}\exp\left(\frac{z^2}{4}\right)\sin\left(\frac{z}{\sqrt{2}}\right) \in S^*,$$

$$J = \frac{1}{3} : w(z) = \sqrt{3}\exp\left(\frac{z^2}{6}\right)\sin\left(\frac{z}{\sqrt{3}}\right) \in S^*\left(\frac{1}{3}\right),$$

$$J = \frac{1}{4} : w(z) = 2\exp\left(\frac{z^2}{8}\right)\sin\left(\frac{z}{2}\right) \in S^*\left(\frac{1}{2}\right),$$

and so on.

3 Main results and their consequences

Theorem 1. Let $P_n(z)$ be non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < J$ ($z \in \mathbb{U};\ J > 0$). Let $w(z)$ be the solution of the initial-value problem:

$$w''(z) + P_n(z)w(z) = 0 \qquad (w(0) = 0;\ w'(0) = 1) \tag{3.1}$$

in $\mathbb{U}$. Then we have

$$1 - J < \operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} < 1 + J \qquad (z \in \mathbb{U}). \tag{3.2}$$

Proof. If we put

$$u(z) = \frac{zw'(z)}{w(z)} - 1 \qquad (z \in \mathbb{U}), \tag{3.3}$$

then $u(z)$ is analytic in $\mathbb{U}$, $u(0) = 0$ and (3.1) becomes

$$[u(z)]^2 + u(z) + zu'(z) = z^2 P_n(z), \tag{3.4}$$

or equivalently

$$h(u(z), zu'(z)) = z^2 P_n(z), \tag{3.5}$$

where $h(r, s) = r^2 + r + s$. It is easy to check $h(r, s) \in \mathcal{H}_J$, *i.e.*
(i) $h(r, s)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$,
(ii) $(0, 0) \in \mathbb{D}$, $|h(0, 0)| = 0 < J$,
(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J \qquad (K \geq J)$.
From assumption, we have

$$|z^2 P_n(z)| < J \qquad (z \in \mathbb{U};\ J > 0).$$

By using Lemma 1, we have

$$|u(z)| < J \qquad (z \in \mathbb{U};\ J > 0),$$

which, in view of the relationship (3.3), yieds

$$\left|\frac{zw'(z)}{w(z)} - 1\right| < J \qquad (z \in \mathbb{U};\ J > 0),$$

that is,

$$1 - J < \operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} < 1 + J \qquad (z \in \mathbb{U};\ J > 0). \tag{3.6}$$

$\square$

Putting $J = 1$ in Theorem 1, we have the following Corollary.
Corollary 1. Let $P_n(z)$ be a non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < 1$ $(z \in \mathbb{U})$. Let $w(z)$ be the solution of the initial-value problem (3.1) in $\mathbb{U}$. Then $w(z)$ is starlike in $\mathbb{U}$.

Remark 3. It is well know that every solution $w(z)$ of the initial-value problem (3.1) is an entire function.

Example 4. Let $P_2(z) = \frac{3}{4} - \frac{z^2}{16}$ in Corollary 1. (n=2), the solution of the following initial-value problem:

$$w''(z) + \left(\frac{3}{4} - \frac{z^2}{16}\right) w(z) = 0 \qquad (w(0) = 0;\ w'(z) = 1) \tag{3.7}$$

is given by

$$w(z) = z \exp\left(-\frac{z^2}{8}\right) \in S^*.$$

$$w''(z) + \left(\frac{6}{7} - \frac{4}{49}z^2\right) w(z) = 0 \implies w(z) = z \exp\left(-\frac{z^2}{7}\right) \in S^*.$$

$$w''(z) + \left(\frac{9}{10} - \frac{9}{100}z^2\right) w(z) = 0 \implies w(z) = z \exp\left(-\frac{3}{20}z^2\right) \in S^*.$$

Remark 4. Let $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (z \in \mathbb{U})$.
If $\sum_{k=0}^{n} |a_k| \leq 1$, then $|P_n(z)| < 1 \quad (z \in \mathbb{U})$.

Theorem 2. Let $P_n(z)$ be a non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < J \quad (z \in \mathbb{U}; 0 < J \leq 1)$. Let $w(z) \quad (z \in \mathbb{U})$ be the solution of the initial-value problem (3.1) in $\mathbb{U}$. Then $w(z)$ is strongy starlike of orderd, that is,

$$\left|\arg\left\{\frac{zw'(z)}{w(z)}\right\}\right| < \frac{\pi}{2}\alpha \tag{3.8}$$

for some $\alpha \quad (0 < \alpha \leq 1)$ and

$$\alpha = \frac{2}{\pi}\sin^{-1} J \qquad (0 < J \leq 1) \tag{3.9}$$

Proof. If we put

$$u(z) = \frac{zw'(z)}{w(z)} - 1 \qquad (z \in \mathbb{U}), \tag{3.10}$$

tehn $u(z)$ is analytic in $\mathbb{U}, u(0) = 0$ and (3.1) becomes

$$[u(z)]^2 + u(z) + zu'(z) = z^2 P_n(z), \tag{3.11}$$

or equivalently

$$h(u(z), zu'(z)) = z^2 P_n(z), \tag{3.12}$$

where $h(r,s) = r^2 + r + s$. It is easy to check $h(r,s) \in \mathcal{H}_J$, that is,
(i) $h(r,s)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;
(ii) $(0,0) \in \mathbb{D}$, $|h(0,0)| = 0 < J$;
(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J \qquad (1 \geq K \geq J > 0)$.

From assmption, we have

$$|z^2 P_n(z)| < J \qquad (z \in \mathbb{U};\ 0 < J \leq 1).$$

By using Theorem E, we obtain

$$|u(z)| < J \qquad (z \in \mathbb{U};\ 0 < J \leq 1).$$

Therefore, we have

$$\left|\arg\left\{\frac{zw'(z)}{w(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for some α $(0 < \alpha \leq 1)$ and $\alpha = \frac{2}{\pi}\sin^{-1} J$ $(0 < J \leq 1)$.

□

Remark 5. Putting $\alpha = 1$ in Theorem 2, we have Corollary 1.
For $a(z) = -z$ and $b(z) = \lambda$ $(\lambda \in \mathbb{C})$ in Theorem D, the initial-value problem (2.13) becomes

$$w''(z) - zw'(z) + \lambda w(z) = 0 \qquad (w(0) = 0;\ w'(0) = 1), \tag{3.13}$$

which, under the following transformation:

$$w(z) = \exp\left(\frac{z^2}{4}\right) v(z), \tag{3.14}$$

assumes the normal form as given below

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) v(z) = 0 \qquad (v(0) = 0;\ v'(z) = 1). \tag{3.15}$$

These differential equations (3.13) and (3.15) are well-known, so called respectively Hermite's differential equation and Weber's differential equation.

Next, we prove

Theorem 3. We consider Weber's differential equation (3.12). If

$$\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < J \qquad (z \in \mathbb{U};\ 0 < J \leq 1), \tag{3.16}$$

then $v(z)$ is strongly starlike of order α, that is,

$$\left|\arg\left\{\frac{zv'(z)}{v(z)}\right\}\right| < \frac{\pi}{2}\alpha \tag{3.17}$$

for some α $(0 < \alpha \leq 1)$, and α is satisfies (3.9).

Proof of Theorem 3 is similar to the proof of Theorem 2. Taking $\alpha = 1$ in Theorem 3, we have

Corollary 2. (See [10]) We consider Weber's differential equation (3.15). If

$$\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < 1 \qquad (z \in \mathbb{U}), \tag{3.18}$$

then the solution $v(z)$ is starlike in $\mathbb{U}$.

We need the following lemma to prove next result.

Lemma 1. (See [4]) Let $h(r,s,t) : \mathbb{C}^3 \longrightarrow \mathbb{C}$ such that
(i) $h(r,s,t)$ is conti. in a domain $\mathbb{D} \subset \mathbb{C}^3$;
(ii) $(0,0,0) \in \mathbb{D}$ and $|h(0,0,0| < J \qquad (J > 0)$;
(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in \mathbb{D}, K \geq J and \mathrm{Re}[Le^{-i\theta}] \geq 0$.
Let $w(z) = w_1 z + w_2 z^2 + \cdots$ be analytic in $\mathbb{U}$. If $(w(z), zw'(z), z^2 w''(z)) \in \mathbb{D} \quad (z \in \mathbb{U})$ and

$$|h(w(z), zw'(z), z^2 w''(z))| < J \qquad (z \in \mathbb{U}), \tag{3.19}$$

then $|w(z)| < J \quad (z \in \mathbb{U})$.

Applying Lemma 1, we prove the following theorem.

Theorem 4. We consider the Weber's differential equation (3.15). Let $|\lambda + \frac{1}{2} - \frac{z^2}{4}| < J \quad (z \in \mathbb{U};\ 0 < J \leq J)$, then we have

$$\left|\arg\left\{\frac{v(z)}{z}\right\}\right| < \frac{\pi}{2}\alpha \tag{3.20}$$

for some $\alpha \quad (0 < \alpha \leq 1)$, and satisfies (3.9).
Proof. We put

$$u(z) = \frac{v(z)}{z} - 1 \qquad (z \in \mathbb{U}) \tag{3.21}$$

Then $u(z)$ is analytic in $\mathbb{U}, u(0) = 0$ and

$$\frac{2zu'(z)}{1+u(z)} + \frac{z^2 u''(z)}{1+u(z)} = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) \tag{3.22}$$

or equivalently

$$h(u(z), zu'(z), z^2 u''(z)) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right), \tag{3.23}$$

where $h(r,s,t) = \frac{2s}{1+r} + \frac{t}{1+r}$.
It is easy to check the following conditions, that is, that
(i) $h(r,s,t)$ is continous in $\mathbb{D} = \mathbb{C} \setminus \{-1\} \times \mathbb{C} \times \mathbb{C}$;
(ii) $(0,0,0) \in \mathbb{D}$ and $|h(0,0,0)| = 0 < J \qquad (0 < J \leq 1)$;
(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in \mathbb{D}$, $1 \geq K \geq J$ and $\mathrm{Re}[Le^{-i\theta}] \geq 0$.

From assumption of Theorem, we have

$$\left|-z^2\left(\lambda+\frac{1}{2}-\frac{z^2}{4}\right)\right| < J \qquad (z \in \mathbb{U};\ 0 < J \leq 1).$$

By using Lemma 1, we obtain

$$|u(z)| < J \qquad (z \in \mathbb{U};\ 0 < J \leq 1). \tag{3.24}$$

Therefore, we have

$$\left|\arg\left\{\frac{v(z)}{z}\right\}\right| < \frac{\pi}{2}\alpha$$

for some α $(0 < \alpha \leq 1)$ and α is satisfies (3.9).

□

Putting $\alpha = 1$ in Theorem 4, we obtain

Corollary 3. We consider the Weber's differential equation (3.15). Let $|\lambda+\frac{1}{2}-\frac{z^2}{4}| < 1$ $(z \in \mathbb{U})$, then we have

$$\mathrm{Re}\left\{\frac{v(z)}{z}\right\} > 0. \tag{3.25}$$

Now, we recall next lemma by Yamaguchi.

Lemma 2. ([12]) Let $f(z) = z + a_2z^2 + \cdots$ be analytic in $\mathbb{U}$. If $\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > 0$ $(z \in \mathbb{U})$, then we have

$$\mathrm{Re}\{f'(z)\} > 0 \tag{3.26}$$

for $|z| < \sqrt{2} - 1$.

Applying Lemma 2, we have the following Corollary.

Corollary 4. We consider the Weber's differential equation (3.15). Let $|\lambda+\frac{1}{2}-\frac{z^2}{4}| < 1$ $(z \in \mathbb{U})$, then $v(z)$ is close-to-convex in $|z| < \sqrt{2} - 1$.

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